ESE 2017

UPSC ENGINEERING SERVICES EXAMINATION

Preliminary Examination

Paper I

General Studies and Engineering Aptitude

3

Engineering Mathematics and Numerical Analysis

Comprehensive Theory *with* Practice Questions

As per new syllabus of ESE 2017

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ESE-2017 : Preliminary Examination
Paper-I : General Studies and Engineering Aptitude

Engineering Mathematics and Numerical Analysis

1st Edition : 2016

Preface

The compilation of this book **Engineering Mathematics and Numerical Analysis** was motivated by the desire to provide a concise book which can benefit students to understand the concepts of engineering mathematics topics.

B. Singh (Ex. IES)

This textbook **Engineering Mathematics and Numerical Analysis** provides all the requirements of the students, i.e. comprehensive coverage of theory, fundamental concepts and objective type questions articulated in a lucid language. The concise presentation will help the readers grasp the theory of this subject with clarity and apply them with ease to solve objective questions quickly. This book not only covers the syllabus of ESE but also addresses the need of many other competitive examinations. Topics like 'Linear Algebra, Calculus, Differential Equations, Complex Functions, Probability & Statistics, Numerical Method and Transform Theory' are given full emphasis, keeping in mind of our research on their importance in competitive examinations.

We have put in our sincere efforts to present detailed theory and MCQs without compromising the accuracy of answers. For the interest of the readers, some notes, do you know and interesting facts are given in the comprehensive manner. At the end of each chapter, sets of practice question are given with their keys, that will allow the readers to evaluate their understanding of the topics and sharper their question solving skills.

Our team has made their best efforts to remove all possible errors of any kind. Nonetheless, we would highly appreciate and acknowledge if you find and share with us any printing and conceptual errors.

It is impossible to thank all the individuals who helped us, but we would like to sincerely thank all the authors, editors and reviewers for putting in their efforts to publish this book.

With Best Wishes

B. Singh

CMD, MADE EASY

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■■■■

2 Calculus

2.1 Limit

2.1.1 Definition

A number A is said to be limit of a function $f(x)$ at $x = a$ if for any arbitrarily chosen positive integer $\in$, however small but not zero there exist a corresponding number δ greater than zero such that: $|f(x) - A| < \in$ for all values of x for which $0 < |x - a| < \delta$ where $|x - a|$ means the absolute value of $(x - a)$ without any regard to sign.

2.1.2 Right and Left Hand Limits

If x approaches a from the right, that is, from larger value of x than a, the limit of f as defined before is called the right hand limit of $f(x)$ and is written as:

$$\underset{x \to a+0}{Lt} f(x) \text{ or } f(a+0) \text{ or } \underset{x \to a^+}{Lt} f(x)$$

Working rule for finding right hand limit is, put $a + h$ for x in $f(x)$ and make h approach zero.

In short, we have, $$f(a+0) = \underset{h \to 0}{Lt} f(a+h)$$

Similarly if x approaches a from left, that is from smaller values of x than a, the limit of f is called the left hand limit and is written as:

$$\underset{x \to a-0}{Lt} f(x) \text{ or } f(a-0) \text{ or } \underset{x \to a^-}{Lt} f(x)$$

In this case, we have, $$f(a-0) = \underset{h \to 0}{Lt} f(a-h)$$

If both right hand and left hand limit of f, as $x \to$ a exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If however, either or both of these limits do not exist, the limit of f as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

$\therefore$ when $$\underset{x \to a^+}{Lt} f(x) = \underset{x \to a^-}{Lt} f(x)$$

then $$\underset{x \to a}{Lt} f(x) = \underset{x \to a^+}{Lt} f(x) = \underset{x \to a^-}{Lt} f(x)$$

Limit of a function can be any real number, ∞ or $-\infty$. It an sometimes be ∞ or $-\infty$, which are also allowed values for limit of a function.

2.1.3 Various Formulae

These formulae are sometimes useful while taking limits.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots..$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots..$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots...$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots...$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots..\ |x|<1$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots..\right)\ |x|<1$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots..$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots..$$

$$\sin h\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots..$$

$$\cos h\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots..$$

Remember: $\log 1 = 0$; $\log e = 1$; $\log \infty = \infty$; $\log 0 = -\infty$

2.1.4 Some Useful Results

1. $\underset{x\to 0}{Lt}\dfrac{\sin x}{x} = 1$
2. $\underset{x\to 0}{Lt}\cos x = 1$
3. $\underset{x\to 0}{Lt}\dfrac{\tan x}{x} = 1$
4. $\underset{x\to 0}{Lt}(1+x)^{\frac{1}{x}} = e$
5. $\underset{x\to 0}{Lt}(1+nx)^{\frac{1}{x}} = e^n$
6. $\underset{x\to \infty}{Lt}\left(1+\dfrac{1}{x}\right)^x = e$
7. $\underset{x\to \infty}{Lt}\left(1+\dfrac{a}{x}\right)^x = e^a$

2.1.5 Indeterminate Forms

A fraction whose numerator and denominator both tend to zero as $x \to a$ is an example of an indeterminate form written as 0/0. It has no definite values. Other indeterminate forms are: ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^0. (Indeterminate form are not any definite number and hence are not acceptable as limits. To find limit in such cases, we use the L'hospital's rule)

2.1.5.1 Indeterminate Form-I $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

Use L'hospital's Rule.

L'Hospital Rule: If $f(x)$ and $\phi(x)$ be two functions of x and if,

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} \phi(x) = 0$$

or if

$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} \phi(x) = \infty,$$

then

$$\underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x \to a}{Lt} \frac{f'(x)}{\phi'(x)}$$

provided, the latter limit exists, finite or infinite.

Working Rule: If the limit of $f(x)/\phi(x)$ as $x \to a$ takes the form 0/0, differentiate the numerator and denominator separately with respect to x and obtain a new function $f'(x)/\phi'(x)$. Now as $x \to a$ if it again takes the form 0/0, differentiate the numerator and denominator again with respect to x and repeat the above process, until the indeterminate form is removed and we get either a real number, ∞ or – ∞ as a limit.

Caution: Before applying L'Hospital's rule at any stage, be sure that the form is 0/0. Do not go on applying this rule, if the form is not 0/0.

2.1.5.2 Indeterminate Form-II ($0 \times \infty$)

This form can be easily reduced to the form 0/0 or to the form ∞/∞, and then L'Hospital's rule may be applied.

$$\text{Let } \underset{x \to a}{\text{Limit}}\ f(x) = 0 \text{ and } \underset{x \to a}{\text{Limit}}\ \phi(x) = \infty.$$

Then we can write

$$\underset{x \to a}{\text{Limit}}\ f(x) \cdot \phi(x) = \underset{x \to a}{\text{Limit}}\ \frac{f(x)}{1/\phi(x)} \text{ [form 0/0] or } \underset{x \to a}{\text{Limit}}\ \frac{\phi(x)}{1/f(x)} \text{ [form } \infty/\infty]$$

Thus $\underset{x \to a}{\text{Limit}}\ f(x) \cdot \phi(x)$ is reduced to the form 0/0 or ∞/∞ which can now be evaluated by L′ Hospital rule.

2.1.5.3 Indeterminate Form-III (0^0 or 1^∞ or ∞^0)

Suppose $\underset{x \to a}{\text{Limit}}\ [f(x)]^{\phi(x)}$ takes any one of these three forms.

Then

$$\text{let } y = \underset{x \to a}{\text{Limit}}\ [f(x)]^{\phi(x)}$$

Taking log on both sides, we get

$$\log y = \underset{x \to a}{\text{Limit}}\ \phi(x) \cdot \log f(x).$$

Now in any of these above cases log y takes the form $0 \times \infty$ which is changed to the form 0/0 or ∞/∞ then it can be evaluated by previous methods.

2.2 Continuity

2.2.1 Definition

A function $f(x)$ is defined for $x = a$ is said to be continuous at $x = a$ if:

1. $f(a)$ i.e., the value of $f(x)$ at $x = a$ is a definite number and
2. the limit of the function $f(x)$ as $x \to a$ exists and is equal to the value of $f(x)$ at $x = a$.

Note: On comparing the definitions of limit and continuity we find that a function $f(x)$ is continuous at $x = a$ if

$$\underset{x \to a}{\text{Limit}}\ f(x) = f(a)$$

Thus $f(x)$ is continuous at $x = a$ if we have $f(a + 0) = f(a - 0) = f(a)$, otherwise it is discontinuous at $x = a$.

2.2.2 Continuity from Left and Continuity from Right

Let f be a function defined on an open interval I and let a be any point in I. We say that f is continuous from the left at a, if $\underset{x \to a-0}{\text{Limit}}\ f(x)$ exists and is equal to $f(a)$. Similarly f is said to be continuous from the right at a, if $\underset{x \to a+0}{\text{Limit}}\ f(x)$ exists and is equal to $f(a)$.

$\therefore$ A function $f(x)$ is continuous at $x = a$, if it is continuous from left as well as continuous from right.

2.2.3 Continuity in an Open Interval

A function f is said to be continuous in open interval (a, b), if it is continuous at each point of open interval.

2.2.4 Continuity in a Closed Interval

Let f be a function defined on the closed interval (a, b) f is said to be continuous on the closed interval $[a, b]$ if it is:

1. continuous from the right at a and
2. continuous from the left at b and
3. continuous on the open interval (a, b).

2.3 Differentiability

Derivative at a point: Let-I denote the open interval (a, b) in R and let $x_0 \in I$. Then a function $f : I \to R$ is said to be differentiable at x_0, if:

$$\underset{h \to 0}{\text{Limit}}\left[\frac{f(x_0 + h) - f(x_0)}{h}\right] \text{ or } \underset{x \to x_0}{\text{Limit}}\left[\frac{f(x) - f(x_0)}{x - x_0}\right]$$

exist (**finitely**) and is denoted by $f'(x_0)$.

2.3.1 Progressive and Regressive Derivatives

The progressive derivative of f (or right derivative of f) at $x = x_0$ is given by

$\underset{h \to 0}{\text{Limit}}\ \dfrac{f(x_0 + h) - f(x_0)}{h}$, $h > 0$ and is denoted by $Rf'(x_0)$ or by $f'(x_0 + 0)$ or by $f'(x_0^+)$.

The regressive derivative of f (or left derivative of f) at $x = x_0$ is given by

$\underset{h \to 0}{\text{Limit}}\ \dfrac{f(x_0 - h) - f(x_0)}{-h}$, $h > 0$ and is denoted by $Lf'(x_0)$ or by $f'(x_0 - 0)$ or by $f'(x_0^-)$.

2.3.2 Differentiability in an Open Interval

A function f is said to be differentiable in an open interval (a, b), if it is differentiable at each point of the open interval.

2.3.3 Differentiability in a Closed Interval

A function $f : [a, b] \rightarrow R$ is said to be differentiable in closed interval $[a, b]$ if it is

1. differentiable from right a t a [i.e. $R f'(a)$ exists] and
2. differentiable from left at b [i.e. $L f'(a)$ exists] and
3. differentiable in the open interval (a, b).

2.3.4 Relationship between Differentiability and Continuity

Theorem: If a function is differentiable at any point, then it is necessarily continuous at that point, proof of this theorem follows from definitions of differentiability and continuity.

Note: The converse of this theorem not true.

i.e. Continuity is a necessary but not a sufficient condition for the existence of a finite derivative (differentiability).

i.e. differentiability $\Rightarrow$ continuity

But continuity $\not\Rightarrow$ differentiability

2.4 Mean Value Theorems

2.4.1 Rolle's Theorem

If a function $f(x)$ is such that:

1. $f(x)$ is continuous in the closed interval $a \le x \le b$ and
2. $f'(x)$ exists for every point in the open interval $a < x < b$ and
3. $f(a) = f(b)$,

   then there exists at least one value of x, say c where $a < c < b$ such that $f'(c) = 0$.

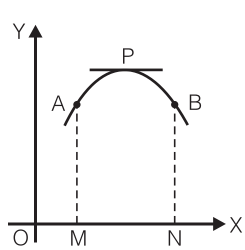
Note: Rolle's theorem will not hold good.

1. If $f(x)$ is discontinuous at some point in the interval $a < x < b$
2. If $f'(x)$ does not exist at some point in the interval $a < x < b$ or
3. If $f(a) \neq f(b)$

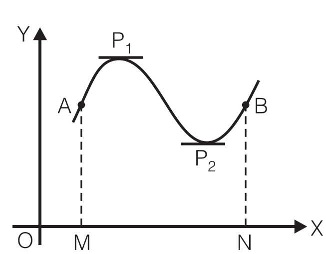
2.4.2 Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers a, b, respectively.

Since $f(x)$ is continuous in $[a, b]$, the curve $y = f(x)$ has a tangent at each point between A and B. Also as $f(a) = f(b)$ the ordinates of the points A and B are equal i.e. $MA = NB$ [See Figure (a)].



(a)



(b)

Then Rolle's theorem asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to x-axis i.e. there exists atleast one real number c in (a, b) such that $f'(c) = 0$. [see figure (a) above]

There may exist more than one point between A and B, the tangents at which are parallel to x-axis [as shown in Figure (b)] i.e. there exists more than one real number c in (a, b) such that $f'(c) = 0$. Rolle's theorem ensures the existence of atleast one real number c in (a, b) such that $f'(c) = 0$.

Remarks:

1. Rolle's theorem fails even if one of the three conditions is not satisfied by the function.
2. The converse of Rolle's theorem is not true, since, $f'(x)$ may be zero at a point in (a, b) without satisfying all the three conditions of Rolle's theorem.

Example 1.

Verify Rolle's theorem for the following functions:

(a) $f(x) = x^2 + x - 6$ in $[-3, 2]$

(b) $f(x) = (x - 1)(x - 2)^2$ in $[1, 2]$

(c) $f(x) = (x^2 - 1)(x - 2)$ in $[-1, 2]$

Solution:

(a) Given $f(x) = x^2 + x - 6$ ... (i)

(i) As $f(x)$ is a polynomial function, it is continuous in $[-3, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(-3, 2)$

(iii) $f(-3) = (-3)^2 - 3 - 6 = 0$, $f(2) = 2^2 + 2 - 6 = 0 \Rightarrow f(-3) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-3, 2)$ such that $f'(x) = 2x + 1$.

Differentiating (i) w.r.t. x, we get $f'(x) = 2x + 1$.

Now $f'(c) = 0 \Rightarrow 2c + 1 = 0 \Rightarrow c = -\frac{1}{2}$.

So there exists $-\frac{1}{2} \in (-3, 2)$ such that $f'\left(-\frac{1}{2}\right) = 0$

Hence, Rolle's theorem is verified.

(b) Given $f(x) = (x - 1)(x - 2)^2$ ... (i)

(i) Since $f(x)$ is a polynomial function, it is continuous is $[1, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(1, 2)$.

(iii) $f(1) = (1 - 1)(1 - 2)^2 = 0$, $f(2) = (2 - 1)(2 - 2)^2 = 0 \Rightarrow f(12) = f(2)$

Thus, all the three conditions of Roll's theorem are satisfied, therefore, there exists atleast one real number c in $(1, 2)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$\begin{aligned} f'(x) &= (x-1)\cdot 2(x-2).\,1 + (x-2)^2.1 \\ &= (x-2)(2x-2+x-2) \\ &= (x-2)(3x-4) \end{aligned}$$

Now $\quad f'(c) = 0$

$\Rightarrow \quad (c-2)(3c-4) = 0$

$\Rightarrow \quad c = 2, 4/3$

But $c \in (1, 2)$, therefore, $c = 4/3$.

So, there exists $(4/3) \in (1, 2)$ such that $f'(4/3) = 0$

Hence, Rolle's theorem is verified.

(c) Given $f(x) = (x^2 - 1)(x - 2)$ ... (i)

(i) Since $f(x)$ is a polynomial function, it is continuous in [–1, 2].

(ii) $f(x)$ being a polynomial function is derivable in (–1, 2).

(iii) $f(-1) = (1 - 1)(1 - 2) = 0$, $f(2) = (4 - 1)(2 - 2) = 0 \Rightarrow f(-1) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in (–1, 2) such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = (x^1 - 1).\ 1 + (x - 2) \cdot 2x = 3x^2 - 4x - 1.$$

Now $$f'(c) = 0 \Rightarrow 3c^2 - 4c - 1 = 0$$

$$\Rightarrow \quad c = \frac{4 \pm \sqrt{16 - 4.3(-1)}}{2.3} = \frac{2 \pm \sqrt{7}}{3}$$

Also $-1 < \frac{2-\sqrt{7}}{3} < \frac{2+\sqrt{7}}{3} < 2 \Rightarrow \frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ both lie in (–1, 2).

So there exist two real numbers $\frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ in (–1, 2) such that

$$f'\left(\frac{2-\sqrt{7}}{3}\right) = 0 \text{ and } f'\left(\frac{2+\sqrt{7}}{3}\right) = 0$$

Hence, Rolle's theorem is verified.

Example 2.

Verify Rolle's theorem for the following functions and find point (or points) where the derivative vanishes:

$$f(x) = \sin x + \cos x \text{ in } \left[0, \frac{\pi}{2}\right]$$

Solution:

Given: $$f(x) = \sin x + \cos x \quad \text{... (i)}$$

(a) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

(b) $f(x)$ is derivable in $\left[0, \frac{\pi}{2}\right]$ and

(c) $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$,

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$

Now $$f'(c) = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow c = 1$$

$$\Rightarrow \quad c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, -\frac{3\pi}{4}, \ldots \text{ but } c \in \left(0, \frac{\pi}{2}\right) \Rightarrow c = \frac{\pi}{4}.$$